

Supersymmetric Euler-Heisenberg effective action: two-loop results

Sergei M. Kuzenko and Simon J. Tyler

School of Physics M013, The University of Western Australia,

35 Stirling Highway, Crawley W.A. 6009, Australia

E-mail: kuzenko@cyllene.uwa.edu.au, styler@physics.uwa.edu.au

ABSTRACT: The two-loop Euler-Heisenberg-type effective action for $\mathcal{N} = 1$ supersymmetric QED is computed within the background field approach. The background vector multiplet is chosen to obey the constraints $D_\alpha W_\beta = D_{(\alpha} W_{\beta)} = \text{const}$, but is otherwise completely arbitrary. Technically, this calculation proves to be much more laborious as compared with that carried out in hep-th/0308136 for $\mathcal{N} = 2$ supersymmetric QED, due to a lesser amount of supersymmetry. Similarly to Ritus' analysis of spinor and scalar QED, the two-loop renormalisation is carried out using proper-time cut-off regularisation. A closed-form expression is obtained for the holomorphic sector of the two-loop effective action, which is singled out by imposing a relaxed super self-duality condition.

KEYWORDS: Supersymmetric Effective Theories, Superspaces, Supersymmetric gauge theory.

Contents

1. Introduction	1
2. Background field setup	2
3. One-loop effective action	4
4. Two-loop quantum corrections	6
5. Renormalisation	10
6. Self-dual background	14
A. Exact superpropagators	16
B. Two-point function and Kähler potential	19

1. Introduction

In the mid-1930s, two nonlinear generalisations of Maxwell's theory were introduced: the Born-Infeld action [1] and the Euler-Heisenberg effective Lagrangian [2] (and its extension for scalar QED [3]). Although these models were soon abandoned by their creators, their impact on the subsequent development of theoretical high-energy physics has been profound. In particular, the Born-Infeld action emerged naturally in string theory [4] (see [5] for a review), as the bosonic sector of the vector Goldstone multiplet action for partial supersymmetry breaking [6], and as an example of self-dual models for nonlinear electrodynamics [7, 8] (see [9] for a review and supersymmetric extensions). As for the effective theories put forward in [2, 3], after Schwinger applied his functional techniques [10] to re-derive and extend the results of [2, 3], the Euler-Heisenberg Lagrangian has become a paradigm for practically all developments related to the evaluation and analysis of low-energy effective actions in quantum field theory, quantum gravity and string theory (for a review of Euler-Heisenberg effective Lagrangians, see [11]).

The one-loop results for spinor and scalar QED [2, 3, 10] were extended in 1975 by Ritus to the two-loop approximation [12]. Further analysis at two loops was carried out by many groups using various techniques, see, e.g., [13–16]. In the supersymmetric case, the two-loop Euler-Heisenberg-type effective action has only been computed for $\mathcal{N} = 2$ supersymmetric QED (SQED) [17], using the covariant supergraph techniques formulated in [18]. The present paper is aimed at extending the results of [17] to the case of $\mathcal{N} = 1$ SQED.

By ‘supersymmetric Euler-Heisenberg action’ we mean the sector of the low energy effective action of the form

$$\Gamma = \frac{1}{e^2} \int d^6z W^2 + \int d^8z W^2 \bar{W}^2 \Omega \left(D^2 W^2, \bar{D}^2 \bar{W}^2 \right). \quad (1.1)$$

Such a functional form is characteristic of the supersymmetric Born-Infeld action [19, 6] and, more generally, self-dual models for nonlinear supersymmetric electrodynamics [9]. To compute the above sector of the effective action within the background field method, it is sufficient to make use of a constant background vector multiplet constrained by

$$D_\alpha W_\beta = D_{(\alpha} W_{\beta)} = \text{const} . \quad (1.2)$$

Since such a vector multiplet is a solution to the equations of motion for any action functional $\Gamma[W_\alpha, \bar{W}_{\dot{\alpha}}]$, the action (1.1) is independent of the choice of gauge fixing in path integral.

This paper is organised as follows. In section 2 we provide the necessary background field setup for $\mathcal{N} = 1$ SQED and, for a special background vector multiplet, express the matter propagators in terms of a single background-dependent Green’s function for which an exact expression is known. The one-loop effective action for $\mathcal{N} = 1$ SQED is reviewed in section 3. Section 4 is the centre of this paper, and is devoted to the evaluation of the two-loop quantum corrections. Renormalisation of the previous sections’ results is discussed in section 5. In section 6 we derive a closed-form expression for a special holomorphic sector of the two-loop effective action. For this paper to be self-contained, we also included two technical appendices. Appendix A contains the expressions for the exact propagators in the presence of a constant background vector multiplet. Appendix B contains a simple derivation of the one-loop Kähler potential and chiral two-point function in the Fermi-Feynman gauge.

2. Background field setup

The classical action for $\mathcal{N} = 1$ SQED is

$$S_{\text{SQED}} = \frac{1}{e^2} \int d^6z W^\alpha W_\alpha + \int d^8z \left(\bar{Q}_+ e^V Q_+ + \bar{Q}_- e^{-V} Q_- \right) + \left(m \int d^6z Q_+ Q_- + \text{c.c.} \right), \quad (2.1)$$

where the gauge field is described by a real unconstrained prepotential V with $W_\alpha = -(1/8)\bar{D}^2 D_\alpha V$ its gauge invariant field strength. The supersymmetric matter is realised in terms of chiral superfields Q_+ and Q_- of charge +1 and -1 respectively.

It is instructive to compare the action (2.1) with that for $\mathcal{N} = 2$ SQED:

$$S_{\text{SQED}}^{\mathcal{N}=2} = \frac{1}{e^2} \int d^8z \bar{\Phi} \Phi + \frac{1}{e^2} \int d^6z W^\alpha W_\alpha + \int d^8z \left(\bar{Q}_+ e^V Q_+ + \bar{Q}_- e^{-V} Q_- \right) + \left(\int d^6z \Phi Q_+ Q_- + \text{c.c.} \right), \quad (2.2)$$

with Φ a neutral chiral superfield. Here the dynamical variables Φ and V realise an abelian $\mathcal{N} = 2$ vector multiplet, while the superfields Q_+ and Q_- constitute a massless Fayet-Sohnius hypermultiplet. The case of a massive hypermultiplet is obtained from (2.2) by the shift $\Phi \rightarrow \Phi + m$.

One can see that the classical action of $\mathcal{N} = 1$ SQED, eq. (2.1), is obtained from (2.2) by discarding Φ as a dynamical variable, and instead ‘freezing’ Φ to a constant value m . This also holds in quantum theory at the one-loop level. Specifically, if $\Gamma_{\mathcal{N}=1}^{(1)}[W]$ and $\Gamma_{\mathcal{N}=2}^{(1)}[W, \Phi]$ are the vector multiplet sectors of the one-loop effective actions for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ SQED, respectively, then they are related to each other as follows: $\Gamma_{\mathcal{N}=1}^{(1)}[W] = \Gamma_{\mathcal{N}=2}^{(1)}[W, m]$. However, this simple correspondence is already broken at two loops, due to the presence of additional supergraphs (involving internal $\Phi\bar{\Phi}$ lines) in the $\mathcal{N} = 2$ case. This has the dramatic implication that the two-loop Euler-Heisenberg-type action for $\mathcal{N} = 1$ SQED is much more difficult to evaluate than the $\mathcal{N} = 2$ case [17].

To quantise the theory (2.1) within the background field formulation we first rewrite the action in terms of gauge covariantly chiral superfields Q_{\pm} and their conjugates.

$$S_{\text{SQED}} = \frac{1}{e^2} \int d^6z W^\alpha W_\alpha + \int d^8z \left(\bar{Q}_+ Q_+ + \bar{Q}_- Q_- \right) + \left(m \int d^6z Q_+ Q_- + \text{c.c.} \right), \tag{2.3}$$

where Q_{\pm} satisfy the constraints $\bar{D}_{\dot{\alpha}} Q_{\pm} = 0$, with the gauge covariant derivatives $\mathcal{D}_A = (D_a, \mathcal{D}_\alpha, \bar{D}^{\dot{\alpha}}) = D_A + iA_A(z)$ obeying the algebra

$$\begin{aligned} \{ \mathcal{D}_\alpha, \mathcal{D}_\beta \} &= \{ \bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}} \} = 0, & \{ \mathcal{D}_\alpha, \bar{D}_{\dot{\beta}} \} &= -2i \mathcal{D}_{\alpha\dot{\beta}}, \\ [\mathcal{D}_\alpha, \mathcal{D}_{\beta\dot{\beta}}] &= 2i \varepsilon_{\alpha\beta} \bar{W}_{\dot{\beta}}, & [\bar{D}_{\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] &= 2i \varepsilon_{\dot{\alpha}\dot{\beta}} W_\beta, \\ [\mathcal{D}_{\alpha\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] &= i \mathcal{F}_{\alpha\dot{\alpha}, \beta\dot{\beta}} = -\varepsilon_{\alpha\beta} \bar{D}_{\dot{\alpha}} \bar{W}_{\dot{\beta}} - \varepsilon_{\dot{\alpha}\dot{\beta}} \mathcal{D}_\alpha W_\beta. \end{aligned} \tag{2.4}$$

Here the action of W_α on Q_{\pm} is defined as follows: $W_\alpha Q_{\pm} = \pm W_\alpha Q_{\pm}$.

In accordance with the $\mathcal{N} = 1$ background field formulation [20, 21], we split the dynamical variables into background and quantum,

$$Q_{\pm} \rightarrow Q_{\pm} + q_{\pm}, \quad \mathcal{D}_\alpha \rightarrow e^{-\hat{e}v} \mathcal{D}_\alpha e^{\hat{e}v}, \quad \bar{D}_{\dot{\alpha}} \rightarrow \bar{D}_{\dot{\alpha}}, \tag{2.5}$$

with lower-case letters used for the quantum superfields. Here \hat{e} is the charge operator, $\hat{e}q_{\pm} = \pm eq_{\pm}$. The quantum matter superfields q_{\pm} are background covariantly chiral,

$$\bar{D}_{\dot{\alpha}} q_{\pm} = 0. \tag{2.6}$$

In this paper, we are mainly interested in the slowly varying part of the effective action that solely depends on the vector multiplet. For this it is sufficient to only consider a background that satisfies

$$\partial_a W_\beta = D^\beta W_\beta = 0, \quad Q_{\pm} = 0. \tag{2.7}$$

Upon quantisation in the Fermi-Feynman gauge, we end up with the following action to be used for loop calculations

$$S_{\text{quantum}} = -\frac{1}{2} \int d^8z v \square v + \int d^8z \left(\bar{q}_+ e^{ev} q_+ + \bar{q}_- e^{-ev} q_- \right) + \left(m \int d^6z q_+ q_- + \text{c.c.} \right). \quad (2.8)$$

From here we can read off the propagators in the standard manner

$$\begin{aligned} i \langle v(z) v(z') \rangle &= -G_0(z, z'), \\ i \langle q_+(z) q_-(z') \rangle &= -m G_+(z, z') = \frac{m}{4} \bar{\mathcal{D}}^2 G(z, z'), \\ i \langle q_+(z) \bar{q}_+(z') \rangle &= G_{+-}(z, z') = \frac{1}{16} \bar{\mathcal{D}}^2 \mathcal{D}^2 G(z, z'), \\ i \langle \bar{q}_-(z) q_-(z') \rangle &= G_{-+}(z, z'). \end{aligned} \quad (2.9)$$

The above matter propagators are expressed via the Green's function $G(z, z')$ which satisfy the equation

$$(\square_v - m^2)G(z, z') = -\delta^8(z - z'), \quad \square_v = \mathcal{D}^a \mathcal{D}_a - \mathcal{W}^\alpha \mathcal{D}_\alpha + \bar{\mathcal{W}}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}}, \quad (2.10)$$

and is characterised by the proper-time representation (A.7) and (A.8). The proper-time representation for the free, massless Green's function, $G_0(z, z')$, which determines the gauge field propagator is

$$\begin{aligned} G_0(z, z') &= i \int_0^\infty ds K_0(z, z'|s) e^{-\epsilon s}, \quad \epsilon \rightarrow +0, \\ K_0(z, z'|s) &= K_{\text{bos}}(\rho|s) \delta^4(\zeta) = -\frac{i}{(4\pi s)^2} e^{i\rho^2/4s} \delta^4(\zeta), \end{aligned} \quad (2.11)$$

with K_{bos} the bosonic heat kernel, and the two point functions $(\rho^\alpha, \zeta^\alpha, \bar{\zeta}_{\dot{\alpha}})$ defined in (A.10). The interactions are easily read from (2.8) by expanding in the quantum fields.

3. One-loop effective action

Although the one-loop Euler-Heisenberg effective action for SQED has been calculated in many other places [22–26, 17], we will repeat it here for the sake of completeness and in order to establish some notation. Its formal representation is (see [27] for an introduction to heat kernel techniques in superspace)

$$\Gamma_{\text{unren}}^{(1)} = -i \text{Tr}_+ \ln G_+ = -i \int_{s_0}^\infty \frac{ds}{s} \text{Tr}_+ K_+(s) e^{-i(m^2 - i\epsilon)s}, \quad (3.1)$$

where we have introduced a proper-time cut-off to regularise UV divergences. We note that the standard $i\epsilon$ prescription (see also eq. (A.7)) is equivalent to having $\text{Im}(s_0) < 0$.

The above functional trace of the chiral heat kernel (A.18) is defined by

$$\text{Tr}_+ K_+(s) = \int d^6 z K_+(z, z|s), \quad (3.2)$$

so the evaluation of $\Gamma^{(1)}$ reduces to finding the coincidence limit of $K_+(z, z'|s)$. In accordance with the results listed in appendix A, eq. (A.12), it follows that

$$\zeta(s)^2 \Big|_{\zeta \rightarrow 0} = 2W^2 \frac{\cos(sB) - 1}{B^2}. \quad (3.3)$$

This factor of W^2 then prevents any further contributions coming from the action of $U(s)$, thus in the coincidence limit the exponential and the parallel displacement propagator go to unity. In the above we have introduced the notation

$$B^2 = \frac{1}{2} \text{tr} N^2, \quad N_\alpha^\beta = D_\alpha W^\beta; \quad \bar{B}^2 = \frac{1}{2} \text{tr} \bar{N}^2, \quad \bar{N}^{\dot{\alpha}}_{\dot{\beta}} = \bar{D}^{\dot{\alpha}} \bar{W}_{\dot{\beta}}, \quad (3.4)$$

and for the on-shell backgrounds that we are using, the above definitions imply

$$\text{tr} N^{2n} = 2B^{2n}, \quad \text{tr} N^{2n+1} = 0, \quad (3.5)$$

formulae that will be repeatedly used in the following section. These objects also appear in the eigenvalues of $F = F_a^b$ which are equal to $\pm\lambda_+$ and $\pm\lambda_-$, where

$$\lambda_{\pm} = \frac{i}{2}(B \pm \bar{B}). \quad (3.6)$$

This then allows the calculation of the determinant

$$\sqrt{\det \left(\frac{2sF}{e^{2sF} - 1} \right)} = \frac{s\lambda_+}{\sinh(s\lambda_+)} \frac{s\lambda_-}{\sinh(s\lambda_-)} = -\frac{s^2}{2} \frac{B^2 - \bar{B}^2}{\cos(sB) - \cos(s\bar{B})}. \quad (3.7)$$

So, after a small amount of algebra to separate off the term that leads to the UV divergence, the heat kernel reduces to

$$K_+(s) = \frac{i}{(4\pi)^2} W^2 \left(1 + \frac{\bar{B}^2(1 - \cos(sB)) - B^2(1 - \cos(s\bar{B}))}{B^2(\cos(sB) - \cos(s\bar{B}))} \right). \quad (3.8)$$

Then, following [26], we note that the quartic and higher order terms on the right of (3.8) contain a factor of \bar{B}^2 , and the latter can be represented as $\bar{B}^2 = \frac{1}{4} \bar{D}^2 \bar{W}^2$ for the background chosen. This allows us to bring the unrenormalised one-loop effective action to the form:

$$\begin{aligned} \Gamma_{\text{unren}}^{(1)} &= \frac{1}{(4\pi)^2} \int_{s_0}^{\infty} \frac{ds}{s} e^{-i(m^2 - i\epsilon)s} \int d^6 z W^2 \\ &+ \frac{1}{(4\pi)^2} \int d^8 z W^2 \bar{W}^2 \int_0^{\infty} ds \frac{\bar{B}^2(\cos(sB) - 1) - B^2(\cos(s\bar{B}) - 1)}{s B^2 \bar{B}^2 (\cos(sB) - \cos(s\bar{B}))} e^{-i(m^2 - i\epsilon)s}. \end{aligned} \quad (3.9)$$

The first term is obviously UV divergent as $s_0 \rightarrow 0$ and is absorbed into the renormalisation of e^2 . This is discussed in detail in section 5.

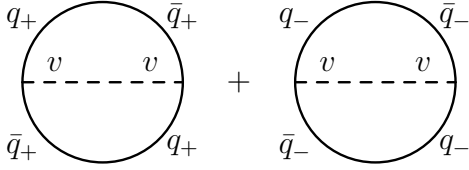


Figure 1: Two-loop supergraph I

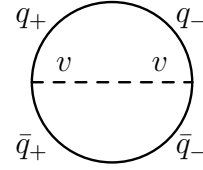


Figure 2: Two-loop supergraph II

4. Two-loop quantum corrections

We now come to the central calculation of this paper, the two-loop quantum correction to the effective action. There are two non-zero,¹ two-loop 1PI supergraphs, as shown in figures 1 and 2.

The first diagram contributes

$$\Gamma_{\text{I}}^{(2)} = \frac{e^2}{2^8} \int d^8 z \int d^8 z' G_0(z, z') \mathcal{D}^2 \bar{\mathcal{D}}^2 G(z, z') \bar{\mathcal{D}}^2 \mathcal{D}^2 G(z', z), \quad (4.1)$$

whilst the contribution from the second diagram is

$$\Gamma_{\text{II}}^{(2)} = -\frac{e^2}{2^4} m^2 \int d^8 z \int d^8 z' G_0(z, z') \bar{\mathcal{D}}^2 G(z, z') \mathcal{D}^2 G(z', z). \quad (4.2)$$

Inserting the proper-time representations for the Green's functions into (4.1) gives

$$\Gamma_{\text{I}}^{(2)} = -ie^2 \int d^8 z d^8 z' \int_0^\infty ds dt du K_0(z, z'|u) K_{-+}(z, z'|s) K_{-+}(z', z|t) e^{-i(m^2 - i\epsilon)(s+t)}, \quad (4.3)$$

and similarly for $\Gamma_{\text{II}}^{(2)}$. We should emphasise that here we only collect the unregulated two-loop quantum corrections. The issues of regularisation and renormalisation will be discussed in detail in the following section.

Before plunging into actual calculations, it is instructive to compare the quantum correction (4.1) with its counterpart in the case of $\mathcal{N} = 2$ SQED [17]. As mentioned in section 2, in the $\mathcal{N} = 2$ case there is a third diagram which can be combined with the first to give a dramatic simplification. Their combined total contribution can be obtained from (4.3) by replacing

$$K_{-+}(z, z'|s) \rightarrow K_{-+}(z, z'|s) - K_{+-}(z, z'|s) = \frac{1}{16} [\mathcal{D}^2, \bar{\mathcal{D}}^2] K(z, z'|s), \quad (4.4)$$

where we have used the identity [18]

$$\bar{\mathcal{D}}'^2 K(z, z'|s) = \bar{\mathcal{D}}^2 K(z, z'|s). \quad (4.5)$$

Then, the structure of the resulting quantum correction is such that $K_{-+}(z', z|t)$ can be equivalently replaced

$$2K_{-+}(z', z|t) \rightarrow K_{-+}(z', z|t) - K_{+-}(z', z|t) = -\frac{1}{16} [\mathcal{D}^2, \bar{\mathcal{D}}^2] K(z', z|t). \quad (4.6)$$

¹There is a third 1PI supergraph, the so-called ‘figure eight’ graph, whose contribution is easily seen to be zero in the Fermi-Feynman gauge.

In the Grassmann coincidence limit, it can be shown [17] that

$$[\mathcal{D}^2, \bar{\mathcal{D}}^2]K(z, z'|s)|_{\zeta=0} \propto W_\alpha \bar{W}^{\dot{\alpha}}, \quad (4.7)$$

and similarly for the expression in (4.6). As a result, the $\mathcal{N} = 2$ counterpart of (4.3) contains a factor of $W^2 \bar{W}^2$ in the integrand. It is this technical property that allows the dramatic simplification of all further calculations [17]. This has no analogue in the case of $\mathcal{N} = 1$ SQED.

So we now continue with the evaluation of $\Gamma_I^{(2)}$ by integrating over the primed Grassmann coordinates with the help of the delta function contained in the vector heat kernel. We also shift the remaining spatial integration variables via the rule $\{x, x'\} \rightarrow \{x, \rho\}$ to yield

$$\Gamma_I^{(2)} = ie^2 \int d^8 z d^4 \rho \int_0^\infty ds dt du K_{\text{bos}}(\rho|u) K_{-+}(z, z'|s) K_{-+}(z', z|t) e^{-i(m^2 - i\epsilon)(s+t)} \Big|_{\zeta=0}. \quad (4.8)$$

We see that to find $\Gamma_I^{(2)}$ we first have to calculate the antichiral-chiral heat kernel, defined below, in the Grassmann coincidence limit $(\zeta_\alpha, \bar{\zeta}_{\dot{\alpha}}) \rightarrow 0$.

As was demonstrated in [17], for the constant, on-shell backgrounds that we are considering, the antichiral-chiral heat kernel can be obtained by taking derivatives of the heat kernel (A.8),

$$\begin{aligned} K_{-+}(z, z'|s) &= \frac{1}{16} \mathcal{D}^2 \bar{\mathcal{D}}'^2 K(z, z'|s) \\ &= \frac{-i}{(4\pi s)^2} \sqrt{\det \left(\frac{2sF}{e^{2sF} - 1} \right)} U(s) e^{\frac{i}{4} \tilde{\rho} F \coth(sF) \tilde{\rho} + R(z, z')} I(z, z'), \end{aligned} \quad (4.9)$$

where

$$R(z, z') = \frac{1}{3} (\zeta^2 \bar{\zeta} \bar{W} - \bar{\zeta}^2 \zeta W) - \frac{i}{2} \tilde{\rho}^a (\mathcal{W} \sigma_a \bar{\zeta} + \zeta \sigma_a \bar{W}) - \frac{i}{12} \tilde{\rho}_{\alpha\dot{\alpha}} (\zeta^\alpha \bar{\zeta} \bar{\mathcal{D}} \bar{W}^{\dot{\alpha}} + 5 \bar{\zeta}^{\dot{\alpha}} \zeta \mathcal{D} W^\alpha),$$

and $\tilde{\rho}$ is a two-point variable that is antichiral in z and chiral in z' ,

$$\tilde{\rho}^a = \rho^a - i \zeta \sigma^a \bar{\zeta}, \quad \mathcal{D}_\beta \tilde{\rho}^a = \bar{\mathcal{D}}'_\beta \tilde{\rho}^a = 0. \quad (4.10)$$

If we use the notation $\Psi(s) \equiv U(s) \Psi U(-s)$ for proper-time dependent variables we see that the action of $U(s)$ in (4.9) can be summarised by modifying the exponential to be

$$e^{\frac{i}{4} \tilde{\rho}(s) F \coth(sF) \tilde{\rho}(s) + R(z, z') + \int_0^s dt (R'(t) + \Xi(t))},$$

where the action of $U(s)$ on ρ , ζ , W^α and $I(z, z')$ is displayed in (A.12) and Ξ is defined in (A.14). The reason for writing $R(s)$ in the convoluted way above becomes clear when we note that $R(z, z')|_{\zeta=0} = 0$ and

$$R'(t) + \Xi(t) = iU(t) \left(2\zeta^2 \bar{W}^2 - \zeta^2 \bar{\zeta} \bar{N} \bar{W} + \frac{i}{2} (\zeta \rho \bar{N} \bar{W} + \zeta N \rho \bar{W}) \right) U(-t). \quad (4.11)$$

It is now a straightforward but tedious task to take the Grassmann coincidence limit. We found it simplest to perform this limit by looking at the first and last term in the exponential separately. Using (3.5) and writing

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \sigma_{\alpha\dot{\alpha}}^a \sigma_{\beta\dot{\beta}}^b F_{ab} = i(\varepsilon_{\alpha\beta} \bar{N}_{\dot{\alpha}\dot{\beta}} + \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{N}_{\alpha\beta}), \quad (4.12)$$

to assist in taking the traces, we get

$$K_{-+}(z, z'|s) \Big|_{\zeta=0} = -\frac{i}{(4\pi s)^2} \det\left(\frac{2sF}{e^{2sF} - 1}\right)^{\frac{1}{2}} \times \exp\left\{\frac{i}{4}\rho F \coth(sF) \rho - iW^\beta \rho_a f_{\beta\dot{\beta}}^a(s) \bar{W}^{\dot{\beta}} - iW^2 \bar{W}^2 f(s)\right\} I(z, z') \Big|_{\zeta=0}, \quad (4.13)$$

where

$$f_{\beta\dot{\beta}}^a(s) = \frac{1}{2}(\mathbb{1} - \coth(sF))_b^a \left(\frac{e^{-isN} - \mathbb{1}}{N} \sigma^b e^{-is\bar{N}} + \sigma^b \frac{e^{-is\bar{N}} - \mathbb{1}}{N}\right)_{\beta\dot{\beta}}, \quad (4.14)$$

$$f(s) = 4 \frac{B \sin(sB) \sin^2(s\bar{B}/2) - \bar{B} \sin(s\bar{B}) \sin^2(sB/2)}{B^2 \bar{B}^2 (\cos(sB) - \cos(s\bar{B}))}.$$

The coincidence limit of $K_{-+}(z', z|t)$ is simply obtained from the above result via the obvious replacements $z \leftrightarrow z'$ and $s \rightarrow t$. Then by pushing the parallel displacement operator through to the left we can combine the two heat kernels to get

$$\Gamma_I^{(2)} = \frac{e^2}{(4\pi)^6} \int d^8 z d^4 \rho \int_0^\infty \frac{ds dt du}{s^2 t^2 u^2} P_+ P_- \times e^{\frac{i}{4}\rho A \rho - iW^\beta \rho_a (f_{\beta\dot{\beta}}^a(s) - f_{\beta\dot{\beta}}^a(t)) \bar{W}^{\dot{\beta}} - iW^2 \bar{W}^2 (f(s) + f(t))} e^{-i(m^2 - i\epsilon)(s+t)}, \quad (4.15)$$

where the parallel displacement operators have annihilated each other, in accordance with (A.11). Here we've introduced the notations

$$P_\pm = \frac{s\lambda_\pm}{\sinh(s\lambda_\pm)} \frac{t\lambda_\pm}{\sinh(t\lambda_\pm)}, \quad (4.16)$$

$$A = F \coth(sF) + F \coth(tF) + \frac{1}{u}, \quad (4.17)$$

where the P_\pm come from the determinant (3.7).

All ρ dependence is now explicit in the exponential, so we can perform the gaussian integral to yield

$$\frac{1}{(4\pi)^2} \int d^4 \rho e^{\frac{i}{4}\rho A \rho - iW^\beta \rho_a (f_{\beta\dot{\beta}}^a(s) - f_{\beta\dot{\beta}}^a(t)) \bar{W}^{\dot{\beta}}} = \frac{i}{\sqrt{\det A}} e^{-iW^2 \bar{W}^2 \mathcal{F}(s, t, u)}, \quad (4.18)$$

where

$$\mathcal{F}(s, t, u) = \frac{1}{4} (f_{\gamma\dot{\gamma}}^a(s) - f_{\gamma\dot{\gamma}}^a(t)) (A^{-1})_a^b (f_b^{\dot{\gamma}\gamma}(s) - f_b^{\dot{\gamma}\gamma}(t)), \quad (4.19)$$

and recalling the eigenvalues of F , (3.6), we obtain

$$\frac{1}{\sqrt{\det A}} = \frac{1}{(a_+ + u^{-1})(a_- + u^{-1})}, \quad (4.20)$$

$$a_{\pm} = \lambda_{\pm} \coth(s\lambda_{\pm}) + \lambda_{\pm} \coth(t\lambda_{\pm}). \quad (4.21)$$

Equation (4.19) can be evaluated with the help of (3.5), (4.12) and the identity

$$(\coth(s\lambda_{\pm}) + 1)(\coth(t\lambda_{\pm}) - 1) = -\frac{e^{i\frac{B\pm\bar{B}}{2}(s-t)}}{\sin(s\frac{B\pm\bar{B}}{2})\sin(t\frac{B\pm\bar{B}}{2})}. \quad (4.22)$$

After some work it yields

$$\mathcal{F}(s, t, u) = \frac{\mathcal{F}_+}{a_+ + u^{-1}} + \frac{\mathcal{F}_-}{a_- + u^{-1}}, \quad (4.23)$$

with $\mathcal{F}_+ \xrightarrow{\bar{B} \rightarrow -\bar{B}} \mathcal{F}_-$ and, taking advantage of the integrands $s \leftrightarrow t$ symmetry,

$$\begin{aligned} \mathcal{F}_+ = & 2 \frac{B^2 \sin^2(\frac{s\bar{B}}{2}) + (B \leftrightarrow \bar{B}) + 2B\bar{B} \cos(s\frac{B+\bar{B}}{2}) \sin(\frac{sB}{2}) \sin(\frac{s\bar{B}}{2})}{B^2 \bar{B}^2 \sin^2(s\frac{B+\bar{B}}{2})} \\ & - 2 \frac{B^2 \cos(B\frac{s-t}{2}) \sin(\frac{s\bar{B}}{2}) \sin(\frac{t\bar{B}}{2}) + (B \leftrightarrow \bar{B}) + 2B\bar{B} \cos(\frac{sB+t\bar{B}}{2}) \sin(\frac{tB}{2}) \sin(\frac{s\bar{B}}{2})}{B^2 \bar{B}^2 \sin(s\frac{B+\bar{B}}{2}) \sin(t\frac{B+\bar{B}}{2})}. \end{aligned} \quad (4.24)$$

Since $W_{\alpha}W_{\beta}W_{\gamma} = 0$ we get a simple, terminating expansion for the remaining exponential in $\Gamma_{\text{I}}^{(2)}$,

$$e^{-iW^2\bar{W}^2(f(s)+f(t)+\mathcal{F}(s,t,u))} = 1 - iW^2\bar{W}^2 \left(f(s) + f(t) + \mathcal{F}(s, t, u) \right).$$

Here the first term does not contribute to the Euler-Heisenberg sector of the effective action (it actually leads to higher derivative quantum corrections), so the first supergraph reduces to

$$\begin{aligned} \Gamma_{\text{I}}^{(2)} = & \frac{e^2}{(4\pi)^4} \int d^8z W^2 \bar{W}^2 \int_0^{\infty} \frac{ds dt du}{s^2 t^2 u^2} \frac{P_+}{a_+ + u^{-1}} \frac{P_-}{a_- + u^{-1}} \\ & \times \left(f(s) + f(t) + \frac{\mathcal{F}_+}{a_+ + u^{-1}} + \frac{\mathcal{F}_-}{a_- + u^{-1}} \right) e^{-i(m^2 - i\epsilon)(s+t)}. \end{aligned} \quad (4.25)$$

The second supergraph is identical to one calculated in [17], so we just restate the result in our notation,

$$\Gamma_{\text{II}}^{(2)} = \frac{e^2}{(4\pi)^4} \int d^8z W^2 \bar{W}^2 \int_0^{\infty} \frac{ds dt du}{s^2 t^2 u^2} \frac{P_+}{a_+ + u^{-1}} \frac{P_-}{a_- + u^{-1}} T(s, t) e^{-i(m^2 - i\epsilon)(s+t)}, \quad (4.26)$$

where

$$T(s, t) = -\frac{8im^2}{B^2\bar{B}^2} \left(\sin^2(\frac{sB}{2}) \sin^2(\frac{t\bar{B}}{2}) + s \leftrightarrow t \right). \quad (4.27)$$

The two proper-time u -integrals in (4.25) and (4.26) can be performed in closed form and are identical to those considered by Ritus [12]. Their direct evaluation gives

$$\int_0^\infty \frac{du}{u^2} \frac{1}{(a_+ + u^{-1})(a_- + u^{-1})} = \frac{1}{a_+ - a_-} \ln \left(\frac{a_+}{a_-} \right), \quad (4.28)$$

$$\begin{aligned} \int_0^\infty \frac{du}{u^2} \frac{1}{(a_+ + u^{-1})(a_- + u^{-1})} \left(\frac{\mathcal{F}_+}{a_+ + u^{-1}} + \frac{\mathcal{F}_-}{a_- + u^{-1}} \right) \\ = \frac{1}{a_+ - a_-} \left(\frac{\mathcal{F}_-}{a_-} - \frac{\mathcal{F}_+}{a_+} \right) + \frac{\mathcal{F}_+ - \mathcal{F}_-}{(a_+ - a_-)^2} \ln \left(\frac{a_+}{a_-} \right). \end{aligned} \quad (4.29)$$

We can now write down the complete unrenormalised 2-loop effective action

$$\begin{aligned} \Gamma_{\text{unren}}^{(2)} = \frac{e^2}{(4\pi)^4} \int d^8z W^2 \bar{W}^2 \int_0^\infty \frac{ds dt}{s^2 t^2} \frac{P_+ P_-}{a_+ - a_-} \left(\frac{\mathcal{F}_-}{a_-} - \frac{\mathcal{F}_+}{a_+} + \right. \\ \left. + \left(f(s) + f(t) + T(s, t) + \frac{\mathcal{F}_+ - \mathcal{F}_-}{a_+ - a_-} \right) \ln \left(\frac{a_+}{a_-} \right) \right) e^{-i(m^2 - i\epsilon)(s+t)}. \end{aligned} \quad (4.30)$$

5. Renormalisation

As previously mentioned, we have regularised the divergences by using a proper-time cut-off. These cut-off dependent divergences are then removed in the standard way, by adding counterterms to the original action. Since the use of the background field method gives us the freedom to rescale the quantum fields [28], and gauge invariance implies that the background gauge field W^α is not renormalised,² the counterterm action takes the simple form

$$\frac{1}{e^2} (Z_e - 1) \int d^6z W^2 + (Z_Q - 1) \left(m \int d^6z q_+ q_- + \text{c.c.} \right). \quad (5.1)$$

We note that the first term above is proportional to the classical action, $\Gamma^{(0)} = \frac{1}{e^2} \int d^6z W^2$. The counterterm coefficients are derived from the multiplicative renormalisation of charge and mass via

$$e^2 = Z_e e_0^2, \quad m^2 = Z_m m_0^2 = Z_Q^{-2} m_0^2, \quad (5.2)$$

where we have used the fact that the $\mathcal{N} = 1$ nonrenormalisation theorem [29, 30, 20] implies that $Z_m^{\frac{1}{2}} Z_Q = 1$. The renormalisation constants are expanded with respect to the fine structure constant, $\alpha = e^2/8\pi$,

$$Z_e = 1 + Z_e^{(1)} + Z_e^{(2)} + \dots, \quad Z_Q = 1 + Z_Q^{(1)} + Z_Q^{(2)} + \dots. \quad (5.3)$$

It is worth noting that in (S)QED an expansion in α is equivalent to the loop expansion.

²Normally it is the combination eV that is renormalisation invariant, but we have absorbed the charge into the field strength.

Each term in the loop expansion of the effective action is constructed from both the standard diagrams computed in the sections above and from diagrams with counterterm insertions. There is a freedom in how much of the finite part of $\Gamma_{\text{unren}}^{(n)}$ is to be removed by the counterterm contribution $\Gamma_{\text{ct}}^{(n)}$. This corresponds to the freedom of choosing the finite part of the charge and matter renormalisation and can be fixed by either choosing a consistent subtraction scheme, for example a (modified) minimal subtraction, or by enforcing some renormalisation conditions.

We choose to work with physical parameters and thus calculate the counterterms using physical renormalisation conditions. Following [31], we define the physical charge squared as the inverse of the coefficient in front of the W^2 term. This clearly leads to the correct charge in the gauge-matter coupling. The physical mass is harder to define from within the Euler-Heisenberg sector of the effective action. The standard way to proceed is to use a separate calculation of, for example, the Kähler potential and use the physical renormalisation conditions in that sector to find the correct mass renormalisation. This is done in appendix B.

First we examine the one-loop renormalisation. Adding the one-loop counterterm contribution to (3.9) yields

$$\Gamma^{(1)} = \Gamma_{\text{ct}}^{(1)} + \Gamma_{\text{unren}}^{(1)} = \frac{1}{e^2} \left(Z_e^{(1)} + \frac{\alpha}{2\pi} E_1(im^2 s_0) \right) \int d^6 z W^2 \tag{5.4}$$

$$+ \frac{1}{(4\pi)^2} \int d^8 z W^2 \bar{W}^2 \int_0^\infty ds \frac{\bar{B}^2 (\cos(sB) - 1) - B^2 (\cos(s\bar{B}) - 1)}{s B^2 \bar{B}^2 (\cos(sB) - \cos(s\bar{B}))} e^{-i(m^2 - i\epsilon)s},$$

where the exponential integral, E_1 , is defined by [32]

$$E_n(z) = \int_1^\infty dt \frac{e^{-zt}}{t^n}, \quad n = 0, 1, 2, \dots, \quad \text{Re}(z) > 0, \tag{5.5}$$

with $E_1(z) = -\ln(ze^\gamma) + O(z)$ where γ is the Euler-Mascheroni constant. It is clear that the renormalisation condition implies

$$Z_e^{(1)} = -\frac{\alpha}{2\pi} E_1(im^2 s_0), \tag{5.6}$$

so that the renormalised one-loop quantum correction is

$$\Gamma^{(1)} = \frac{1}{(4\pi)^2} \int d^8 z W^2 \bar{W}^2 \int_0^\infty ds \frac{\bar{B}^2 (\cos(sB) - 1) - B^2 (\cos(s\bar{B}) - 1)}{s B^2 \bar{B}^2 (\cos(sB) - \cos(s\bar{B}))} e^{-i(m^2 - i\epsilon)s}. \tag{5.7}$$

Now we examine the two-loop renormalisation. The two-loop counterterm contributions, read from (5.1), are

$$\Gamma_{\text{ct}}^{(2)} = Z_e^{(2)} \Gamma^{(0)} + i m^2 Z_Q^{(1)} \left(\text{Tr}_+ G_+ + \text{Tr}_- G_- \right). \tag{5.8}$$

This can be reduced to a more useful form by noting $\text{Tr}_+ G_+ = \text{Tr}_- G_-$, see [27], and that

$$\frac{\partial}{\partial m^2} \Gamma_{\text{unren}}^{(1)} = -i \frac{\partial}{\partial m^2} \int_0^\infty \frac{ds}{s} \text{Tr}_+ K_+ e^{-i(m^2 - i\epsilon)s} = i \text{Tr}_+ G_+ . \quad (5.9)$$

Then using $\Gamma_{\text{unren}}^{(1)} = \Gamma^{(1)} - \Gamma_{\text{ct}}^{(1)}$ combined with the fact

$$m^2 \frac{\partial}{\partial m^2} E_1(im^2 s_0) = -e^{-im^2 s_0} ,$$

we have

$$\Gamma_{\text{ct}}^{(2)} = \left(Z_e^{(2)} - \frac{\alpha}{\pi} Z_Q^{(1)} e^{-im^2 s_0} \right) \Gamma^{(0)} + 2Z_Q^{(1)} m^2 \frac{\partial}{\partial m^2} \Gamma^{(1)} . \quad (5.10)$$

A close examination of the proper-time integrand in the unrenormalised two-loop effective action (4.30) shows that the only divergences that occur are in the $f(s)$ and $f(t)$ terms when t or s go to zero respectively. Writing the unrenormalised result as

$$\Gamma_{\text{unren}}^{(2)} = \frac{e^2}{(4\pi)^4} \int d^8 z W^2 \bar{W}^2 \int_{s_0}^\infty ds dt F(s, t) e^{-i(m^2 - i\epsilon)(s+t)} , \quad (5.11)$$

we can separate off the divergent contribution by adding and subtracting the limit

$$\tilde{F}(s) \equiv \lim_{t \rightarrow 0} t F(s, t) = -2 \frac{B^2 - \bar{B}^2}{B^2 \bar{B}^2} \frac{B \sin(sB) \sin^2(s\bar{B}/2) - (B \leftrightarrow \bar{B})}{(\cos(sB) - \cos(s\bar{B}))^2} \quad (5.12)$$

and similarly for $\tilde{F}(t)$, to give

$$\begin{aligned} \Gamma_{\text{unren}}^{(2)} &= \frac{e^2}{(4\pi)^4} \int d^8 z W^2 \bar{W}^2 \int_0^\infty ds dt \left(F(s, t) - \frac{\tilde{F}(s)}{t} - \frac{\tilde{F}(t)}{s} \right) e^{-i(m^2 - i\epsilon)(s+t)} \\ &\quad + 2E_1(im^2 s_0) \frac{e^2}{(4\pi)^4} \int d^8 z W^2 \bar{W}^2 \int_0^\infty ds \tilde{F}(s) e^{-i(m^2 - i\epsilon)s} . \end{aligned} \quad (5.13)$$

Then, motivated by the form of $\Gamma_{\text{ct}}^{(2)}$ and by previous renormalisations of two-loop Euler-Heisenberg effective actions we note that

$$\frac{1}{(4\pi)^2} \int d^8 z W^2 \bar{W}^2 \int_0^\infty ds \tilde{F}(s) e^{-i(m^2 - i\epsilon)s} = m^2 \frac{\partial}{\partial m^2} \Gamma^{(1)} . \quad (5.14)$$

We can now combine $\Gamma_{\text{unren}}^{(2)}$ with $\Gamma_{\text{ct}}^{(2)}$ and choose Z_Q so that the two-loop effective action is finite (of course, there is still freedom in choosing a finite part). The renormalisation condition then fixes Z_e , yielding

$$Z_Q^{(1)} = -\frac{\alpha}{2\pi} E_1(im^2 s_0) , \quad Z_e^{(2)} = \frac{\alpha}{\pi} Z_Q^{(1)} e^{-im^2 s_0} . \quad (5.15)$$

Thus we see that mass renormalisation at one-loop affects the two-loop charge renormalisation. We note that when calculating with bare parameters and multiplicative renormalisation the mass renormalisation contributes to the charge renormalisation through the simple relation

$$\ln(m_0^2) = \ln(m^2) - \ln(Z_m) . \tag{5.16}$$

Since we are using an ‘on-shell’ renormalisation [13, 33] the appropriate renormalisation equation is the Callan-Symanzik equation [34]. The renormalisation group functions are defined by

$$\beta_{\text{CS}} = m \frac{d\alpha}{dm} = \alpha \frac{d \ln Z_e}{d \ln m} , \quad \gamma_m = \frac{d \ln Z_m^{-\frac{1}{2}}}{d \ln m} . \tag{5.17}$$

In QED it can be shown [13, 33] that the β -function for dimensional regularisation with minimal subtraction coincides with the above β -function to $O(\alpha^3)$. The proofs given also hold for SQED. Inspired by (5.16) we note that to first order $Z_Q - 1 \approx -\frac{1}{2} \ln Z_m$, so to first order in Z_m we can write the charge renormalisation constant as

$$Z_e \approx 1 + \frac{\alpha}{2\pi} (\ln(i e^\gamma m^2 s_0) - \ln Z_m) . \tag{5.18}$$

It is then a simple calculation to get

$$\beta_{\text{CS}} = \frac{\alpha^2}{\pi} (1 + \gamma_m) + O(\alpha^3) , \quad \gamma_m = \frac{\alpha}{\pi} + O(\alpha^2) . \tag{5.19}$$

These results coincide with the known β and γ functions, e.g. [31]. Given that only the one-loop effective action contributes directly to the F^2 term [31, 35] it must be that all higher contributions to the charge renormalisation are due to the mass renormalisation. Therefore, following the arguments of [31], we expect that (5.18) and thus the β -function are exact results.

Finally we can write the renormalised effective action to two loops,

$$\begin{aligned} \Gamma[W, \bar{W}] &= \Gamma^{(0)} + \Gamma^{(1)} + \Gamma^{(2)} = \frac{1}{e^2} \int d^6 z W^2 \\ &+ \frac{1}{(4\pi)^2} \int d^8 z W^2 \bar{W}^2 \int_0^\infty ds \frac{\bar{B}^2 (\cos(sB) - 1) - B^2 (\cos(s\bar{B}) - 1)}{s B^2 \bar{B}^2 (\cos(sB) - \cos(s\bar{B}))} e^{-i(m^2 - i\epsilon)s} \\ &+ \frac{e^2}{(4\pi)^4} \int d^8 z W^2 \bar{W}^2 \int_0^\infty ds dt \left(F(s, t) - \frac{\tilde{F}(s)}{t} - \frac{\tilde{F}(t)}{s} \right) e^{-i(m^2 - i\epsilon)(s+t)} , \end{aligned} \tag{5.20}$$

where B is now understood as

$$B^2 = \frac{1}{4} D^2 W^2 , \tag{5.21}$$

and the vector multiplet is not subject to any constraints. This is the final form of our main result, and using it allows one to compute, by standard means, quantities of interest, such as the vacuum non-persistence amplitude [10, 12].

6. Self-dual background

In this section we will examine the self-dual limit of the Euler-Heisenberg effective action calculated above. Six years ago in [36, 37], it was noted that for both the one and two-loop Euler-Heisenberg effective actions in scalar and spinor QED the proper-time integrals could be fully integrated when the background field is self-dual. The results of the proper-time integrals can be written completely in terms of the function

$$\xi(x) = \frac{1}{2} \int_0^\infty ds \left(\frac{1}{s^2} - \frac{1}{\sinh^2 s} \right) e^{-2xs} = -x \left(\psi(x) - \ln x + \frac{1}{2x} \right), \quad (6.1)$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function [32].

As discussed in [38] and references therein, the effective action for a supersymmetric theory becomes trivial in the case of a self-dual background. Yet we can still impose a relaxed form of self-duality which allows us to retain a holomorphic-like sector of the effective action. If we write the full Euler-Heisenberg effective action as in eq. (1.1) and impose the relaxed self-duality conditions

$$W_\alpha \neq 0, \quad D_\alpha W_\beta = 0, \quad \bar{D}_{\dot{\alpha}} \bar{W}_{\dot{\beta}} = \bar{D}_{(\dot{\alpha}} \bar{W}_{\dot{\beta})} \neq 0 \quad (6.2)$$

then we can track the following sector

$$\int d^8 z W^2 \bar{W}^2 \Omega(0, \bar{D}^2 \bar{W}^2).$$

It should be noted that although the conditions (6.2) are inconsistent with the structure of a single, real vector multiplet, their use is perfectly justified as long as we realise we are only calculating the above term. At the end of the calculation we can remove the self-duality condition and have a well defined sector of the effective action. Since we already have the full two-loop Euler-Heisenberg effective action, we can simply take its limit as $D^2 W^2 \rightarrow 0$ to obtain the above sector.

Taking the self-dual limit of the one-loop effective action we get

$$\Gamma_{\text{SD}}^{(1)} = \frac{x^2}{(4\pi)^2} \int d^8 z \frac{W^2 \bar{W}^2}{m^4} \int_0^\infty ds s \left(\frac{1}{s^2} - \frac{1}{\sinh^2 s} \right) e^{-2sx},$$

where we have rescaled s to be dimensionless and written the field strength in terms of $x = m^2/\bar{B}$, a natural dimensionless variable. As is shown in [38], this can then be expressed in terms of the first derivative of ξ ,

$$\Gamma_{\text{SD}}^{(1)} = -\frac{1}{(4\pi)^2} \int d^8 z \frac{W^2 \bar{W}^2}{m^4} x^2 \xi'(x). \quad (6.3)$$

We will split the two-loop effective action into parts, writing

$$\Gamma_{\text{SD}}^{(2)} = \frac{e^2}{(4\pi)^4} \int d^8 z \frac{W^2 \bar{W}^2}{m^4} (I_H + I_f + I_{\mathcal{F}}). \quad (6.4)$$

The first term, I_{II} , is the contribution from figure 2, and is calculated, as in [38], to be

$$I_{II} = \frac{1}{3} (1 + x^2 \xi'''(x)) . \quad (6.5)$$

The second term is a bit more difficult, being generated by

$$\begin{aligned} I_f &= 2m^4 \int_0^\infty ds dt \left(\frac{P_+ P_-}{a_+ - a_-} \frac{f(s)}{s^2 t^2} \ln \left(\frac{a_+}{a_-} \right) - \frac{1}{t} \tilde{F}(s) \right) e^{-im^2(s+t)} \Big|_{B \rightarrow 0} \\ &= 4x^2 \int_0^\infty ds dt \frac{s^2}{\sinh^2 s} \left(\coth s - \frac{1}{s} \right) \left(\coth(s+t) - \coth t + \frac{1}{t} \right) e^{-2x(s+t)} , \end{aligned} \quad (6.6)$$

where we have used the identity

$$\frac{\sinh t}{\sinh s \sinh(s+t)} = \coth s - \coth(s+t) . \quad (6.7)$$

Part of the above integral for I_f factorises and is easily computed, to yield

$$I_f = -2x^2 \int_0^\infty ds dt \coth(s+t) \frac{d}{ds} \left(\frac{s^2}{\sinh^2 s} \right) e^{-2x(s+t)} - 2x^2 \xi(x) \xi''(x) .$$

The entangled term can then be simplified by repeated integration by parts to give

$$I_f = 2x \xi(x) - 2x^2 (\xi'(x) + \xi(x) \xi''(x)) - x^3 \xi'''(x) . \quad (6.8)$$

The final term is generated by

$$\begin{aligned} I_{\mathcal{F}} &= m^4 \int_0^\infty ds dt \frac{P_+ P_-}{s^2 t^2} \frac{1}{a_+ - a_-} \left(\frac{\mathcal{F}_-}{a_-} - \frac{\mathcal{F}_+}{a_+} + \frac{\mathcal{F}_+ - \mathcal{F}_-}{a_+ - a_-} \ln \left(\frac{a_+}{a_-} \right) \right) e^{-im^2(s+t)} \Big|_{B \rightarrow 0} \\ &= \frac{1}{6} (1 + x^2 \xi'''(x)) + 2x^2 \int_0^\infty \frac{ds dt}{\sinh^2(s+t)} \left(\frac{s^2}{\sinh^2 s} - \frac{st \cosh(s+t)}{\sinh s \sinh t} \right) e^{-2x(s+t)} , \end{aligned} \quad (6.9)$$

where we have separated off a term proportional to I_{II} .

We have not been able to compute the remaining double integral analytically, however using some techniques standard to experimental mathematics, e.g. [39], we have been able to deduce its solution.³ The trick consists of two parts. First we use high precision numerical integration to evaluate the integral for small integer values of x corresponding to large field strengths, $\bar{B} \sim m^2$. We then use the hypothesis that the self-dual effective action can always be reduced to a combination of derivatives of $\xi(x)$ functions with polynomial coefficients. For any particular value of x these derivatives form an independent set of transcendental numbers. We can then use lattice reduction⁴ to find the simplest coefficients

³SJT would like to thank Dr Paul Abbott for suggesting the following method and for supplying the initial routine for identifying transcendental numbers.

⁴Mathematica implements the Lenstra-Lenstra-Lovasz [40] algorithm of lattice reduction.

that match our numerical integral. Doing this for a few values of x was enough to deduce the functional form of $I_{\mathcal{F}}$, the result being

$$I_{\mathcal{F}} = \frac{1}{6} (1 + x^2 \xi'''(x)) + \frac{1}{4} + x^2 - x^3 \xi''(x) - (\xi(x) - x \xi'(x) + x)^2 . \quad (6.10)$$

This result can be checked by comparing its asymptotic expansion with the series expansion of the double integral for weak, self-dual fields using

$$\int_0^\infty ds dt \frac{s^n t^m}{(s+t)^l} e^{-2x(s+t)} = \frac{n! m! (n+m-l+1)!}{(n+m+1)! (2x)^{n+m+2-l}}, \quad l \leq n+m+1 . \quad (6.11)$$

This check is trivial when using computer algebra and has been done to the 100th order in field strength.

We now can write the full two-loop self-dual effective action as

$$\Gamma_{\text{SD}}^{(2)} = \frac{e^2}{(4\pi)^4} \int d^8 z \frac{W^2 \bar{W}^2}{m^4} \left(\frac{1}{2} (1 + x^2 \xi'''(x)) - 2x^3 \xi''(x) + \frac{1}{4} - 2x^2 \xi(x) \xi''(x) - (\xi(x) - x \xi'(x))^2 \right) , \quad (6.12)$$

where we note the first term is just the $\mathcal{N} = 2$ self-dual effective action. From here it is easy to read off expansions for both the weak ($x \gg 1$) and strong ($x \ll 1$) field limits using [32]

$$\xi(x) = \sum_{n=1}^\infty \frac{B_{2n}}{2n x^{2n-1}}, \quad x \gg 1 \quad (6.13)$$

$$\xi(x) = \frac{1}{2} + x(\gamma + \ln x) - \sum_{n=2}^\infty \zeta(n) (-x)^n, \quad x \ll 1, \quad (6.14)$$

where B_n are the Bernoulli numbers and γ is the Euler-Mascheroni constant. This allows for a much simpler deduction of the strong field asymptotics and the particle creation rate than from the effective action (5.20).

Acknowledgments

We are grateful to Joseph Buchbinder for reading the manuscript and useful comments. One of us (S.J.T.) is grateful to Ian McArthur and Paul Abbott for useful discussions. The work of S.M.K. is supported in part by the Australian Research Council. The work of S.J.T. is supported by an Australian Postgraduate Award.

A. Exact superpropagators

In this appendix we review, following [18], the structure of the exact superpropagators in a constant $\mathcal{N} = 1$ abelian vector multiplet background. We start with the gauge covariant derivative algebra defined in (2.4) where the field strengths satisfy the Bianchi identities

$$\bar{D}_{\dot{\alpha}} \mathcal{W}_{\alpha} = \mathcal{D}_{\alpha} \bar{\mathcal{W}}_{\dot{\alpha}} = 0, \quad \mathcal{D}^{\alpha} \mathcal{W}_{\alpha} = \bar{D}_{\dot{\alpha}} \bar{\mathcal{W}}^{\dot{\alpha}} . \quad (\text{A.1})$$

The three major d'Alembertians that occur in covariant supergraphs [21] are the vector, chiral and antichiral d'Alembertians, defined by:

$$\begin{aligned}
 \square_v &= \mathcal{D}^a \mathcal{D}_a - \mathcal{W}^\alpha \mathcal{D}_\alpha + \bar{\mathcal{W}}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}}, \\
 \square_+ &= \mathcal{D}^a \mathcal{D}_a - \mathcal{W}^\alpha \mathcal{D}_\alpha - \frac{1}{2} (\mathcal{D}^\alpha \mathcal{W}_\alpha), \quad \square_+ \Phi = \frac{1}{16} \bar{\mathcal{D}}^2 \mathcal{D}^2 \Phi, \quad \bar{\mathcal{D}}_{\dot{\alpha}} \Phi = 0, \\
 \square_- &= \mathcal{D}^a \mathcal{D}_a + \bar{\mathcal{W}}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} + \frac{1}{2} (\bar{\mathcal{D}}_{\dot{\alpha}} \bar{\mathcal{W}}^{\dot{\alpha}}), \quad \square_- \bar{\Phi} = \frac{1}{16} \mathcal{D}^2 \bar{\mathcal{D}}^2 \bar{\Phi}, \quad \mathcal{D}_\alpha \bar{\Phi} = 0.
 \end{aligned} \tag{A.2}$$

The operators \square_+ and \square_- are related to each other as follows:

$$\mathcal{D}^2 \square_+ = \square_- \mathcal{D}^2, \quad \bar{\mathcal{D}}^2 \square_- = \square_+ \bar{\mathcal{D}}^2, \tag{A.3}$$

whilst for an on-shell background we get the additional, important relations

$$\mathcal{D}^\alpha \mathcal{W}_\alpha = 0 \implies \mathcal{D}^2 \square_+ = \mathcal{D}^2 \square_v = \square_v \mathcal{D}^2, \quad \bar{\mathcal{D}}^2 \square_- = \bar{\mathcal{D}}^2 \square_v = \square_v \bar{\mathcal{D}}^2. \tag{A.4}$$

In what follows, the background vector multiplet is chosen to be covariantly constant and on-shell,

$$\mathcal{D}_a \mathcal{W}_\alpha = 0, \quad \mathcal{D}^\alpha \mathcal{W}_\alpha = 0. \tag{A.5}$$

Associated with the d'Alembertian \square_v is the propagator $G(z, z')$ satisfying the equation

$$(\square_v - m^2) G(z, z') = -\delta^8(z - z'). \tag{A.6}$$

It has the proper-time representation

$$G(z, z') = i \int_0^\infty ds K(z, z'|s) e^{-i(m^2 - i\epsilon)s}, \quad \epsilon \rightarrow +0. \tag{A.7}$$

With the corresponding heat kernel [18]

$$K(z, z'|s) = -\frac{i}{(4\pi s)^2} \sqrt{\det \left(\frac{2s\mathcal{F}}{e^{2s\mathcal{F}} - 1} \right)} U(s) \zeta^2 \bar{\zeta}^2 e^{\frac{i}{4}\rho \mathcal{F} \coth(s\mathcal{F}) \rho} I(z, z'), \tag{A.8}$$

where the determinant is computed with respect to the Lorentz indices,

$$U(s) = \exp(-is(\mathcal{W}^\alpha \mathcal{D}_\alpha - \bar{\mathcal{W}}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}})), \tag{A.9}$$

and the supersymmetric two-point functions $\zeta^A(z, z') = (\rho^a, \zeta^\alpha, \bar{\zeta}_{\dot{\alpha}})$ are defined as follows:

$$\rho^a = (x - x')^a - i\theta\sigma^a\bar{\theta}' + i\theta'\sigma^a\bar{\theta}, \quad \zeta^\alpha = (\theta - \theta')^\alpha, \quad \bar{\zeta}_{\dot{\alpha}} = (\bar{\theta} - \bar{\theta}')_{\dot{\alpha}}. \tag{A.10}$$

$I(z, z')$ is the $\mathcal{N} = 1$ parallel displacement propagator described in [18]. The only properties we need for this calculation are

$$I(z, z') I(z', z) = I(z, z) = 1. \tag{A.11}$$

Introducing the notation for proper-time dependent variables $\Psi(s) \equiv U(s)\Psi U(-s)$ the action of $U(s)$ on the objects appearing in the right hand side of (A.8) is⁵

$$\begin{aligned} \mathcal{W}^\alpha(s) &= (\mathcal{W}e^{-is\mathcal{N}})^\alpha, & \zeta^\alpha(s) &= \zeta^\alpha + \left(\mathcal{W}\frac{e^{-is\mathcal{N}} - 1}{\mathcal{N}}\right)^\alpha, \\ \bar{\mathcal{W}}_{\dot{\alpha}}(s) &= (\bar{\mathcal{W}}e^{is\bar{\mathcal{N}}})_{\dot{\alpha}}, & \bar{\zeta}_{\dot{\alpha}}(s) &= \bar{\zeta}_{\dot{\alpha}} + \left(\bar{\mathcal{W}}\frac{e^{is\bar{\mathcal{N}}} - 1}{\bar{\mathcal{N}}}\right)_{\dot{\alpha}}, \\ \rho_{\alpha\dot{\alpha}}(s) &= \rho_{\alpha\dot{\alpha}} - 2\int_0^s dt(\mathcal{W}_\alpha(t)\bar{\zeta}_{\dot{\alpha}}(t) + \zeta_\alpha(t)\bar{\mathcal{W}}_{\dot{\alpha}}(t)), \\ I(z, z'|s) &= \exp\left\{\int_0^s dt \Xi(\zeta, \mathcal{W}, \bar{\mathcal{W}}|t)\right\} I(z, z'), \end{aligned} \tag{A.12}$$

where

$$\mathcal{N}_\alpha{}^\beta = \mathcal{D}_\alpha\mathcal{W}^\beta, \quad \bar{\mathcal{N}}^{\dot{\alpha}}{}_{\dot{\beta}} = \bar{\mathcal{D}}^{\dot{\alpha}}\bar{\mathcal{W}}_{\dot{\beta}}, \tag{A.13}$$

and

$$\begin{aligned} \Xi(\zeta, \mathcal{W}, \bar{\mathcal{W}}) &= \frac{1}{12}\rho^{\dot{\alpha}\alpha}(\mathcal{W}^\beta\bar{\zeta}^{\dot{\beta}} - \zeta^\beta\bar{\mathcal{W}}^{\dot{\beta}})(\varepsilon_{\beta\alpha}\bar{\mathcal{D}}_{\dot{\beta}}\bar{\mathcal{W}}_{\dot{\alpha}} - \varepsilon_{\dot{\beta}\alpha}\mathcal{D}_\beta\mathcal{W}_\alpha) - \frac{2i}{3}\zeta\mathcal{W}\bar{\zeta}\bar{\mathcal{W}} \\ &\quad - \frac{i}{3}\zeta^2\left(\bar{\mathcal{W}}^2 - \frac{1}{4}\bar{\zeta}\bar{\mathcal{D}}\bar{\mathcal{W}}^2\right) - \frac{i}{3}\bar{\zeta}^2\left(\mathcal{W}^2 - \frac{1}{4}\zeta\mathcal{D}\mathcal{W}^2\right). \end{aligned} \tag{A.14}$$

Associated with the chiral d'Alembertian \square_+ is the propagator $G_+(z, z')$ satisfying the equation

$$(\square_+ - m^2)G_+(z, z') = -\delta_+(z - z'), \quad \delta_+(z - z') = -\frac{1}{4}\bar{\mathcal{D}}^2\delta^8(z - z'). \tag{A.15}$$

It is covariantly chiral in both arguments,

$$\bar{\mathcal{D}}_{\dot{\alpha}}G_+(z, z') = \bar{\mathcal{D}}'_{\dot{\alpha}}G_+(z, z') = 0 \tag{A.16}$$

and for on-shell backgrounds, $\mathcal{D}^\alpha\mathcal{W}_\alpha = 0$, it is related to $G(z, z')$ via

$$G_+(z, z') = -\frac{1}{4}\bar{\mathcal{D}}^2G(z, z') = -\frac{1}{4}\bar{\mathcal{D}}'^2G(z, z'). \tag{A.17}$$

The corresponding heat kernel is

$$K_+(z, z'|s) = -\frac{i}{(4\pi s)^2}\sqrt{\det\left(\frac{2s\mathcal{F}}{e^{2s\mathcal{F}} - 1}\right)} U(s) \zeta^2 e^{\frac{i}{4}\rho\mathcal{F}\coth(s\mathcal{F})} \rho^{-\frac{i}{2}}\rho^\alpha\mathcal{W}\sigma_\alpha\bar{\zeta} I(z, z'). \tag{A.18}$$

It is equivalent to the kernels originally computed in [22, 41]. An antichiral Green's function can be similarly constructed.

The other Green's function that occurs in our Feynman rules is the antichiral-chiral propagator G_{-+} . It plays a central role in our calculations, and is described in section 4.

⁵It should be noted that unlike [17], the index contractions in spinorial matrix functions will always be done in the 'natural' positions.

B. Two-point function and Kähler potential

The Euler-Heisenberg action constitutes a slowly varying part of the full effective action for $\mathcal{N} = 1$ SQED, which depends on the vector multiplet only. Another important sector of the effective action is the chiral matter action $\Gamma[Q, Q^\dagger]$ which is singled out by switching off the gauge field. For completeness, we discuss here the one-loop quantum correction to $\Gamma[Q, Q^\dagger]$ within the background field method. More specifically, we concentrate on evaluating the two-point function and the Kähler potential. Either of these results can then be used to find the one-loop renormalisation of the chiral fields.

Starting from the classical action (2.3) we perform the background-quantum splitting

$$V \rightarrow ev, \quad Q_\pm \rightarrow Q_\pm + q_\pm. \quad (\text{B.1})$$

Then, introducing the matrix notation

$$\hat{v} = \sigma_3 v, \quad \hat{m} = \sigma_1 m, \quad q^T = (q_+, q_-), \quad (\text{B.2})$$

and the gauge invariant quantities

$$M_v^2 = e^2 Q^\dagger Q, \quad \kappa = e^2 Q^T \sigma_1 Q, \quad \bar{\kappa} = e^2 Q^\dagger \sigma_1 \bar{Q}, \quad (\text{B.3})$$

the resulting quadratic quantum action takes the form

$$S_{\text{quad}} = -\frac{1}{2} \int d^8 z v (\square - M_v^2) v + \int d^8 z (q^\dagger q + e q^\dagger \hat{v} Q + e Q^\dagger \hat{v} q) + \frac{1}{2} \left(\int d^6 z q^T \hat{m} q + \text{c.c.} \right),$$

where we have chosen the Fermi-Feynman gauge.

The mixing terms in S_{quad} can be eliminated from the path integral by implementing the shift

$$q(z) \rightarrow q(z) - e \int d^8 z' (G_{++}(z, z') \hat{v}(z') Q(z') + G_{+-}(z, z') \hat{v}(z') Q(z')), \quad (\text{B.4})$$

where

$$\mathbf{G} = \begin{pmatrix} G_{++} & G_{+-} \\ G_{-+} & G_{--} \end{pmatrix} = \begin{pmatrix} \hat{m} \frac{\bar{D}^2}{4} & \frac{\bar{D}^2 D^2}{16} \mathbb{1} \\ \frac{D^2 \bar{D}^2}{16} \mathbb{1} & \hat{m} \frac{D^2}{4} \end{pmatrix} G, \quad (\text{B.5})$$

$$\begin{aligned} G(z, z') &= -\frac{1}{\square - m^2} \delta^8(z, z') \\ &= \frac{1}{(4\pi)^2} \int_0^\infty \frac{ds}{s} e^{\frac{i}{4s} \rho^2 - is(m^2 - i\epsilon)} \delta^4(\zeta) = G_{\text{bos}}(z, z') \delta^4(\zeta), \end{aligned} \quad (\text{B.6})$$

are the chiral fields propagators. The corresponding Jacobian is obviously equal to unity. Then, the quadratic action turns into

$$S_0 = -\frac{1}{2} \int d^8 z v (\square - M_v^2 - \Delta) v + \int d^8 z q^\dagger q + \frac{1}{2} \left(\int d^6 z q^T \hat{m} q + \text{c.c.} \right), \quad (\text{B.7})$$

where the operator Δ , coming from the above field redefinition, is

$$\Delta(z, z') = e^2 \left(Q^\dagger(z) \frac{\bar{D}^2 D^2}{16} G(z, z') Q(z') + Q^T(z) \frac{D^2 \bar{D}^2}{16} G(z, z') \bar{Q}(z') \right. \\ \left. - Q^\dagger(z) \hat{m} \frac{\bar{D}^2}{4} G(z, z') \bar{Q}(z') - Q^T(z) \hat{m} \frac{D^2}{4} G(z, z') Q(z') \right). \quad (\text{B.8})$$

Since the components of \mathbf{G} in (B.5) are background independent, the one-loop effective action is calculated purely from the gauge field's Hessian,

$$\Gamma_{\text{unren}}^{(1)} = \frac{i}{2} \text{Tr} \ln (\square - M_v^2 - \Delta). \quad (\text{B.9})$$

In our first approximation we want to discard all terms that are more than quadratic in the chiral background. This can be achieved by expanding the logarithm to first order,

$$\Gamma_{\text{unren}}^{(1)} \approx \frac{i}{2} \text{Tr} \left(\ln(\square) - \frac{1}{\square} M_v^2 - \frac{1}{\square} \Delta \right). \quad (\text{B.10})$$

Due to a lack of spinor derivatives to annihilate the Grassmann delta function, the first two terms above evaluate to zero. Similarly the last two terms in Δ also do not contribute. This leaves

$$\Gamma_{\text{unren}}^{(1)} \approx \frac{i}{2} e^2 \int d^8 z d^8 z'' G_0(z, z'') \left(Q^\dagger(z'') G_{\text{bos}}(z'', z') Q(z') + Q^T(z'') G_{\text{bos}}(z'', z') \bar{Q}(z') \right) \Big|_{z=z'}$$

where the Green's function G_0 is defined in (2.11), and G_{bos} is the bosonic part of (B.6). Using the proper-time representation for the Green's functions we get

$$\Gamma_{\text{unren}}^{(1)} \approx -ie^2 \int d^8 z d^4 \rho Q^\dagger(z) Q(z') \int_0^\infty \frac{ds dt}{s^2 t^2} e^{i \frac{s+t}{4st} \rho^2 - is(m^2 - i\epsilon)} \Big|_{\theta=\theta'}. \quad (\text{B.11})$$

We note that the above expression involves a single Grassmann integral, although it is non-local in space-time, in accordance with the $\mathcal{N} = 1$ non-renormalisation theorem. It is not difficult to check that this is equivalent to the standard momentum space representation for the two-point function, see, e.g., [21, 45].

To compute the Kähler potential, it suffices to choose Q and Q^\dagger to be constant, and then Δ reduces to

$$\Delta(x, y) = -\frac{1}{\square - m^2} \left(\frac{1}{16} M_v^2 \{D^2, \bar{D}^2\} - m\kappa \frac{D^2}{4} - m\bar{\kappa} \frac{\bar{D}^2}{4} \right) \delta^8(z, z'), \quad (\text{B.12})$$

where κ and $\bar{\kappa}$ are defined in (B.3) above. The effective action is then

$$\Gamma_{\text{unren}}^{(1)} = \frac{i}{2} \text{Tr} \ln \left(1 + \frac{1}{16} \frac{M_v^2 \{D^2, \bar{D}^2\} - 4m\kappa D^2 - 4m\bar{\kappa} \bar{D}^2}{(\square - m^2)(\square - M_v^2)} \right). \quad (\text{B.13})$$

The logarithm can then be factorised using

$$1 + X \bar{D}^2 D^2 + Y D^2 \bar{D}^2 + Z D^2 + \bar{Z} \bar{D}^2 = (1 + N D^2)(1 + U \bar{D}^2 D^2 + V D^2 \bar{D}^2)(1 + \bar{N} \bar{D}^2) \\ N = (1 + 16 \square Y)^{-1} Z, \quad V = Y, \quad U = X - \bar{Z}(1 + 16 \square Y)^{-1} Z,$$

for constant, matrix coefficients. Evaluating the trace in the standard way, by going to momentum space, final result for the Kähler potential is

$$K_{\text{unren}}^{(1)} = -\frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \ln \left(\frac{(k^2(k^2 + m^2) + m^2 M_v^2)^2 + k^2 m^2 \kappa \bar{\kappa}}{k^4 (k^2 + M_v^2)^2} \right). \quad (\text{B.15})$$

This can be compared with the calculation given in [42]. Although we can factorise the above quartic in k^2 and thus perform the momentum integration, it is not very enlightening. On the other hand, in the massless limit the result is greatly simplified. Upon renormalisation at a non-zero field strength Q_0 , we get the familiar result (see, e.g., [43])

$$K^{(1)} \Big|_{m=0} = \frac{e^2}{(4\pi)^2} Q^\dagger Q \left(\ln \frac{Q^\dagger Q}{Q_0^\dagger Q_0} - 2 \right). \quad (\text{B.16})$$

Its functional form is similar to the one-loop Kähler potential for the Wess-Zumino model first computed in [44].

To renormalise the matter sector we only need the quadratic part of the Kähler potential. This can be obtained by either setting $Q(z') \approx Q(z)$ in (B.11) or by expanding the logarithm in (B.15). Choosing the latter course gives

$$K_{\text{unren}}^{(1)} = iM_v^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{k^2 + m^2} + O(|Q|^4) = -\frac{M_v^2}{(4\pi)^2} \int_{s_0}^{\infty} \frac{ds}{s} e^{-is(m^2 - i\epsilon)} + O(|Q|^4),$$

where we have introduced a proper-time cut-off. Enforcing the physical renormalisation condition

$$\frac{\partial^2 K}{\partial Q^\dagger \partial Q} \Big|_{Q=0} = 1, \quad (\text{B.17})$$

yields the matter renormalisation constant

$$Z_Q = 1 - \frac{\alpha}{2\pi} E_1(is_0 m^2) + O(\alpha^2) = 1 + Z_Q^{(1)} + O(\alpha^2). \quad (\text{B.18})$$

Clearly, the expression for $Z_Q^{(1)}$ just obtained coincides with that derived in section 5, eq. (5.15), on the basis of the two-loop renormalisation of the Euler-Heisenberg action.

References

- [1] M. Born and L. Infeld, *Foundations of the new field theory*, *Proc. Roy. Soc. Lond. A* **144** (1934) 425.
- [2] W. Heisenberg and H. Euler, *Consequences of Dirac's theory of positrons*, *Z. Phys.* **98** (1936) 714 [English translation: [physics/0605038](#)].
- [3] V. Weisskopf, *Über die Elektrodynamik des Vakuums auf Grund der Quantentheorie des Elektrons*, *Kong. Dans. Vid. Selsk. Math-fys. Medd. XIV* **6** (1936) 3, reprinted in *Quantum Electrodynamics*, J. Schwinger (Ed.), Dover, New York, (1958); English translation: in A.I. Miller, *Early quantum electrodynamics*, Cambridge University Press, Cambridge (1994).
- [4] E.S. Fradkin and A.A. Tseytlin, *Nonlinear electrodynamics from quantized strings*, *Phys. Lett.* **B 163** (1985) 123.

- [5] A.A. Tseytlin, *Born-Infeld action, supersymmetry and string theory*, in *The many faces of the superworld*, M. Shifman (Ed.), World Scientific (2000) [hep-th/9908105](#).
- [6] J. Bagger and A. Galperin, *A new Goldstone multiplet for partially broken supersymmetry*, *Phys. Rev. D* **55** (1997) 1091 [[hep-th/9608177](#)].
- [7] G.W. Gibbons and D.A. Rasheed, *Electric-magnetic duality rotations in nonlinear electrodynamics*, *Nucl. Phys. B* **454** (1995) 185 [[hep-th/9506035](#)].
- [8] M.K. Gaillard and B. Zumino, *Self-duality in nonlinear electromagnetism*, in *Supersymmetry and Quantum Field Theory*, J. Wess and V. P. Akulov (Eds.), Springer (1998), [hep-th/9705226](#).
- [9] S.M. Kuzenko and S. Theisen, *Nonlinear self-duality and supersymmetry*, *Fortschr. Phys.* **49** (2001) 273 [[hep-th/0007231](#)].
- [10] J. S. Schwinger, *On gauge invariance and vacuum polarization*, *Phys. Rev.* **82** (1951) 664.
- [11] G.V. Dunne, *Heisenberg-Euler effective Lagrangians: basics and extensions*, in *From fields to strings: circumnavigating theoretical physics*, M. Shifman, A. Vainshtein and J. Wheeler (Eds.), World Scientific, Singapore (2005), [hep-th/0406216](#).
- [12] V.I. Ritus, *The Lagrange function of an intensive electromagnetic field and quantum electrodynamics at small distances*, *Sov. Phys. JETP* **42** (1975) 774 [*Zh. Eksp. Teor. Fiz.* **69** (1975) 1517].
- [13] W. Dittrich and M. Reuter, *Effective Lagrangians in quantum electrodynamics*, Springer, Berlin (1985).
- [14] M. Reuter, M.G. Schmidt and C. Schubert, *Constant external fields in gauge theory and the spin 0, 1/2, 1 path integrals*, *Ann. Phys. (NY)* **259** (1997) 313 [[hep-th/9610191](#)].
- [15] D. Fliegner, M. Reuter, M.G. Schmidt and C. Schubert, *Two-loop Euler-Heisenberg Lagrangian in dimensional regularization*, *Theor. Math. Phys.* **113** (1997) 1442 [[hep-th/9704194](#)].
- [16] B. K rs and M.G. Schmidt, *The effective two-loop Euler-Heisenberg action for scalar and spinor QED in a general constant background field*, *Eur. Phys. J. C* **6** (1999) 175 [[hep-th/9803144](#)].
- [17] S.M. Kuzenko and I.N. McArthur, *Low-energy dynamics in $N = 2$ super QED: two-loop approximation*, *JHEP* **10** (2003) 029 [[hep-th/0308136](#)].
- [18] S.M. Kuzenko and I.N. McArthur, *On the background field method beyond one loop: a manifestly covariant derivative expansion in super Yang-Mills theories*, *JHEP* **05** (2003) 015 [[hep-th/0302205](#)].
- [19] S. Cecotti and S. Ferrara, *Supersymmetric Born-Infeld Lagrangians*, *Phys. Lett. B* **187** (1987) 335.
- [20] M.T. Grisaru, W. Siegel and M. Ro ek, *Improved methods for supergraphs*, *Nucl. Phys. B* **159** (1979) 429.
- [21] S.J. Gates, M.T. Grisaru, M. Ro ek and W. Siegel, *Superspace, or one thousand and one lessons in supersymmetry*, *Front. Phys.* **58** (1983) 1 [[hep-th/0108200](#)].
- [22] K. Shizuya and Y. Yasui, *Construction of effective actions in superspace*, *Phys. Rev. D* **29** (1984) 1160.

- [23] T. Ohrndorf, *The effective Lagrangian of supersymmetric Yang-Mills theory*, *Phys. Lett. B* **176** (1986) 421.
- [24] I.N. McArthur and T.D. Gargett, *A *Gaussian* approach to computing supersymmetric effective actions*, *Nucl. Phys. B* **497** (1997) 525 [[hep-th/9705200](#)].
- [25] N.G. Pletnev and A.T. Banin, *Covariant technique of derivative expansion of one-loop effective action*, *Phys. Rev. D* **60** (1999) 105017 [[hep-th/9811031](#)].
- [26] I.L. Buchbinder, S.M. Kuzenko and A.A. Tseytlin, *On low-energy effective actions in $N = 2,4$ superconformal theories in four dimensions*, *Phys. Rev. D* **62** (2000) 045001 [[hep-th/9911221](#)].
- [27] I.L. Buchbinder and S.M. Kuzenko, *Ideas and methods of supersymmetry and supergravity or a walk through superspace*, IOP, Bristol (1998).
- [28] L.F. Abbott, *The background field method beyond one loop*, *Nucl. Phys. B* **185** (1981) 189; *Introduction to the background field method*, *Acta Phys. Polon.* **13** (1982) 33.
- [29] J. Wess and B. Zumino, *A Lagrangian model invariant under supergauge transformations*, *Phys. Lett. B* **49** (1974) 52; *Broken supergauge symmetry and renormalization*, *Nucl. Phys. B* **76** (1974) 310;
S. Ferrara, J. Iliopoulos and B. Zumino, *Supergauge invariance and the Gell-Mann-Low eigenvalue*, *Nucl. Phys. B* **77** (1974) 413;
R. Delbourgo, *Superfield perturbation theory and renormalization*, *Nuovo Cim.* **A25** (1975) 646.
- [30] P.C. West, *The supersymmetric effective potential*, *Nucl. Phys. B* **106** (1976) 219;
S. Weinberg, *Ambiguous solutions of supersymmetric theories (F1)*, *Phys. Lett. B* **62** (1976) 111.
- [31] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Exact Gell-Mann-Low function in supersymmetric electrodynamics*, *Sov. Phys. JETP* **42** (1985) 224 [*Phys. Lett. B* **166** (1986) 334]; *An exact relation for the Gell-Mann-Low function in supersymmetric electrodynamics*, *Phys. Lett. B* **166** (1986) 334.
- [32] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover (1964);
The Wolfram Functions Site at functions.wolfram.com.
- [33] R. Coquereaux, *Dimensional renormalization and comparison of renormalization schemes in quantum electrodynamics*, *Ann. Phys. (NY)* **125** (1980) 401.
- [34] C.G. Callan Jr., *Broken scale invariance in scalar field theory*, *Phys. Rev. D* **2** (1970) 1541;
K. Symanzik, *Small distance behavior in field theory and power counting*, *Commun. Math. Phys.* **18** (1970) 227.
- [35] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Exact Gell-Mann-Low function of supersymmetric Yang-Mills theories from instanton calculus*, *Nucl. Phys. B* **229** (1983) 381; *Beta function in supersymmetric gauge theories: instantons versus traditional approach*, *Phys. Lett. B* **166** (1986) 329;
M.T. Grisaru, B. Milewski and D. Zanon, *The structure of UV divergences in SSYM theories*, *Phys. Lett. B* **155** (1985) 357.
- [36] G.V. Dunne and C. Schubert, *Closed-form two-loop Euler-Heisenberg Lagrangian in a self-dual background*, *Phys. Lett. B* **526** (2002) 55 [[hep-th/0111134](#)].

- [37] G.V. Dunne and C. Schubert, *Two-loop self-dual Euler-Heisenberg Lagrangians. I: real part and helicity amplitudes*, *JHEP* **08** (2002) 053 [[hep-th/0205004](#)]; *Two-loop self-dual Euler-Heisenberg Lagrangians. II: imaginary part and Borel analysis*, *JHEP* **06** (2002) 042 [[hep-th/0205005](#)].
- [38] S.M. Kuzenko and I.N. McArthur, *Relaxed super self-duality and effective action*, *Phys. Lett. B* **591** (2004) 304 [[hep-th/0403082](#)].
- [39] D.H. Bailey, J.M. Borwein, N.J. Calkin, R. Girgensohn, D.R. Luke and V. Moll, *Experimental mathematics in action*, A.K. Peters, ISBN 978-1-56881-271-7 (2006).
- [40] A.K. Lenstra, H.W. Lenstra and L. Lovasz, *Factoring polynomials with rational coefficients*, *Math. Ann.* **261** (1982) 515.
- [41] I. L. Buchbinder and S. M. Kuzenko, *Calculation of the superpropagator in supersymmetric quantum electrodynamics in a constant external field*, *Russ. Phys. J.* **28** (1985) 61.
- [42] M.T. Grisaru, M. Roček and R. von Unge, *Effective Kähler potentials*, *Phys. Lett. B* **383** (1996) 415 [[hep-th/9605149](#)].
- [43] A. Pickering and P.C. West, *The one loop effective super-potential and non-holomorphicity*, *Phys. Lett. B* **383** (1996) 54 [[hep-th/9604147](#)].
- [44] I.L. Buchbinder, S. Kuzenko and Z. Yarevskaya, *Supersymmetric effective potential: superfield approach*, *Nucl. Phys. B* **411** (1994) 665.
- [45] P. West, *Introduction to supersymmetry and supergravity*, World Scientific, Singapore (1990).